Facilitation Guide

Module 1: What is Argumentation?

This module is the first of five modules created for professional learning purposes as part of the Bridging Math Practices project. An Overview for our facilitation guides and the modules is available at http://bridges.education.uconn.edu/argumentation-pd-modules/ . This module can be used independently or in conjunction with one or more of the other four modules. We encourage user to become familiar with the set of materials and then adapt them to your particular needs and timeframe.

This Facilitation Guide includes the following:

All handouts and other materials for Module 1 can be found at http://bridges.education.uconn.edu/what-is-a-mathematical-argument/

Goals: Module 1

- Develop a deeper understanding of argumentation and its potential in the math classroom.
- Analyze mathematical arguments using the concepts of claim, evidence, and warrants.
- Establish community agreements and expectations for a positive professional working environment.

Overarching Questions for 5-Module Sequence

- What is a mathematical argument? What "counts" as an argument?
- What is the purpose(s) of argumentation in mathematics? In the math classroom?
- How do we organize our classroom to support student participation in the practice of mathematical argumentation, and to support them in developing their proficiency with argumentation (both verbal/interactive and written forms)?
- What does student argumentation look like at different levels of proficiency?

Background Information: Mathematical Argumentation

Argumentation is perhaps the core of mathematics. It is through mathematical argumentation that mathematicians develop, refine, revise, refute, vet, and ultimately establish something that was thought to be true as true.

A defining feature of a mathematical argument is the fact that it is produced with the aim of demonstrating a claim true or false. Likewise, a defining feature of mathematical argumentation is a process that aims to sort out and ultimately establish a (well-defined) claim and the supporting logic by which the claim can be shown to be true.

Despite argumentation's centrality to math, there is no one shared definition of mathematical argumentation. It encompasses a range of practices. You can see more detail on this idea in the narrated slides available at http://wp.msp.education.uconn.edu/wp-content/uploads/sites/1351/2015/06/IntroductoryVideo-What-is-a-mathematical-argument2.pptx.

Argumentation is different from proof in that a mathematical *proof* is generally the final, clean up argument, and it adheres to a certain level of formality or rigor that is often not associated with the idea of a *mathematical argument*. The general process of *proving*, that considers all the activities that go into developing and establishing a conjecture, is more akin to *mathematical argumentation*, though some find it challenging to take a term associated with "proof" and cast it so broadly. See the Additional Resources for Module 1 at the end of this guide for some references that focus on mathematical proof and proving.

Argumentation, in our view, is different from *justification* in that one can offer a justification for more than the truth of a claim. One can offer a justification for a host of different decisions, such as one's choice of method and one's conjectures. Justification then is not limited to determining the truth (or falsehood) of a mathematical claim. For example, if a student decides to use the quadratic formula instead of factoring to solve a quadratic, one might ask the student to justify that decision – *why did you choose to use the quadratic formula?* There's no "truth" involved – it was a choice – so the question asks for justification of that decision and not a mathematical argument to show the decision was "true" or "correct."[1] That said, there is much overlap between justification and argumentation. As there are no agreed upon definitions, the terminology you decide is ultimately a choice you/your school can make, ideally supported by discussions so you create shared meanings around these terms.

Materials: Module 1

Copies of handouts
Powerpoint slides (draft slides provided)
Projection capability
Sentence strips
Tape
Markers
Large chart paper with reproduced student work samples for Micah, Angel, Roland, and Kira (optional)

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[1] If, however, the quadratic cannot be solved using factoring, one could offer an argument to demonstrate that that claim is true. An argument for this claim might begin: *This quadratic cannot be solving using factoring because….*

Timing and Activity Table for Module 1

Session activity and focus	Estimated Timing		Materials
	Monthly (1.5 hrs)	Workshop (3.5 hrs)	
Opening Activities: Participants and/or facilitators take time to get to know one another	10 min	20 min[2]	*variable, depending on activity*
Activity 1.1 Community Agreements: Establish ground rules for how the group will interact and collaborate.	20 min	25 min	Markers, Sentence Strips, Tape Handout 1: *Community Agreements*
Activity 1.2 Sum of Consecutive Numbers Task: Participants engage with *Sum of Two Consecutive Numbers Task*. They work through task individually, producing an argument, and then discuss in small and whole groups.	15 min	20 min	Handout 2: *Sum of Consecutive Numbers Task.*
Activity 1.3 Student Work Samples – Consecutive Sums Task: The general structure of an argument is introduced. Participants describe and analyze sample responses to the consecutive numbers task in small groups. The whole group debriefs the activity.	35 min	35 min	Handout 3: *Consecutive Sums Student Work*
Activity 1.4 Analyze Components of Arguments: Introduce structure of an argument using language claim, warrant, and evidence. Work through one shared argument to illustrate components. Revisit student work samples. Participants wrap up this segment by reflecting on their original argument.	*Combined with 1.3 in this format*	25 min	Optional: provide as a handout the slide on the Structure of an Argument with the language *claims, warrant,* and *evidence.* (slide 22 of the draft slides) Handout 4: *Reflecting on Argument Quality*
Activity 1.5 Bridging to Practice: For monthly PLC format, discuss the work to be completed between sessions	7 min	70 min	Monthly: (facilitator-created) Handout(s) for Bridging-to-Practice Activity

[2] For this module, we allocate 20 minutes of our 3.5 hours of materials to an ice breaker. We include the Opening Activity in the timing for this first module only. In general, this Opening Activity is not part of the 3.5 hours of materials for the Workshop Format. We anticipate that facilitators will use Opening Activity as appropriate for their groups and the actual time they have with their participants.

For Workshop Format: Participants engage in a Student Argumentation Work Sample Sorting Activity focused on the quality of students' written arguments.			Workshop: (i) Handout: *Student Work Sorting Protocol;* (ii) Student work samples sorting packet for each individual; (iii) Student work samples sorting packet - one for each group [Full set of Work Sample packets available at http://bridges.education.uconn.edu/argumentation-resource-packets-2/]
Activity 1.6 Session Closure: Reflect on the day and/or administer a feedback survey	3 min	15 min	

Implementation Guide and Possibilities: Module 1

In the sections that follow we provide suggestions on how to use the materials for two different models of professional development: monthly meetings during the school year and an intensive five-day workshop. We also include the goals of specific activities (indicating how they contribute to the goals of the module) and some of our reasoning for including particular activities and/or materials. Following each activity description, we include a table with common issues for the different activities and suggest questions or prompts you might use to help address those issues.

Opening Activity:

In our enactments of the materials, we utilized two forms of icebreaker activities at the beginning of the first session. We describe each here. These specific ice breakers are not connected with the module content, so please feel free to use any ice breaker you like.

Sweet Links: This activity assumes you do not want particular groupings during this module. It is also best suited for the Workshop Format given the timing.

This ice breaker activity can be used to help distribute teachers across small groups in an entertaining and somewhat random fashion, and to highlight the many similar motivations and challenges participants shared as they embarked in this new PD experience together.

Depending on your numbers, place four different types of candy bars (any size, or use another colorful item) on each table. Ask participants to choose one piece of candy from the set available at each small group table. Then ask participants to regroup based on the candy they selected, for example, have all of the teachers who pick "SweetTarts" move to sit together, and those teachers who pick "Twix Bars" sit together, etc. It helps if tables are numbered. Hidden, on the back side of the table numbers, record the candy group name. Then when it is time, the table numbers can be turned over to reveal candy name and participants can move to their new candy groups.

Once regrouped, introduce yourself, and provide some brief personal facts (e.g., I taught high school algebra, I have one son). Others listen to the facts, and see where they can "link up" to the facilitator by sharing something in common. When a participant hears a fact s/he can link to, they raise their hand, join you in the front of the room by linking arms, and explain the link. Then the newly standing participant introduces himself/herself and the process repeats until everyone in the room is "linked up." The premise of the activity is that all of the participants and facilitator(s) share important connections.

Find-Someone-Who Ice Breaker: Find-Someone-Who is a fun activity for finding commonalities among participants in the room. It also gets participants up and moving.

Give each participant a handout card (all cards are the same). We have included an example in the handout *1-OpeningActivities_Find-Someone*. Explain that they have 5 minutes (or whatever time you like) to fill in as many squares as they can, prioritizing getting 4 in a row. To "fill in" a square, participants put the initials of the person in the square they find who matches the descriptor.

After the time is up, you can ask who has one "bingo," two "bingos," etc. and/or identify the participant with the most squares filled in. Those participants with the bingos or the most squares can read the names of the people they filled in, and share the relevant information. This activity supports the goal of getting to know more about the participants in the room, and perhaps find some otherwise hidden connections.

Activity 1.1 Community Agreements

In our work with groups, we have found that it is important to set ground rules for how the group will interact. Establishing expectations early helps create an environment where participants feel comfortable and respected sharing their opinions and grappling with new ideas. It also creates space for having a conversation about how the group is working together in future sessions.

To facilitate this Community Agreements activity, ask participants to think silently about their answers to the question: *What are some things that are important for a group to agree to in order for that group to work well together?* Allow about two minutes of silent brainstorming and jotting down thoughts. Then ask participants to share their ideas with their small group. The small group considers the ideas, refining them into a set of ideas they feel are important to share with the whole group for inclusion in the set of community agreements.

Each small group can either share one idea, rotating around the groups, or share all their ideas. The groups' ideas are offered to the whole group for consideration. It is important for you as the facilitator of the activity to make space for questions about the meaning of the proposed agreement, or refinements to the agreements.

Once accepted by the whole group, a designated reporter/recorder for the proposing small group records those accepted agreements onto a sentence strip (one agreement per sentence strip) and these are posted to make them public to the whole group. (These can remain there for the week, if doing the Workshop Format, and we also suggest that the set of Community Agreements be typed up to share in word form with the group at the next meeting.)

After each of the small groups posts their sentence strips, you can acknowledge that revisions, adjustments or additions can be made as necessary throughout the course of the professional development experience to help the group better accomplish their work together. Below is an example of a set of community agreements created by one of our cohorts.

Be Present. Eliminate Distractions
Be Positive
Have a Common Goal
Actively Listen and Participate
Stay Engaged
Contribute
Respect Support the Diverse
Opinions in the Room
Show Respect by Not Interrupting
Give Constructive Feedback

Activity 1.2 Sum of Two Consecutive Numbers Task

This activity requires participants to produce a mathematical argument (written), and analyze student arguments. The purposes of this activity are to (a) show a range of types of arguments, or approaches to producing and argument, and discuss the viability of those approaches; (b) solidify a defining purpose of an argument, namely to demonstrate the claim is true; and (c) begin to develop participants' analytic skills reviewing and critiquing the strengths and weaknesses of arguments.[3]

The slide presentation preceding the activity introduces participants to a general definition of mathematical argumentation and sets the stage for the ideas discussed throughout the

[3] Original problem from Knuth, E., & Sutherland, J. (2004). Student understanding of generality. In D. McDougall & J. Ross (Eds.), *Proceedings of the 26th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Toronto, Canada: University of Ontario* (Vol 2., pp 561-568).

rest of the professional development modules. You may wish to also provide slides 13 and/or 16 as handouts to participants, as this can be a useful reminder and reference regarding arguments. As a part of this conversation, we highlighted the ways that mathematical argumentation fits into the Standards of Mathematical Practice established by the *Common Core State Standards for Mathematics* in general and the Connecticut Core Standards specifically. In particular, engaging students in argumentation aligns fully with mathematical practice 3: *Construct viable arguments and critique the reasoning of others.* Engaging students in mathematical argumentation also provides opportunities for students to develop practices 1 and 6, *Make sense of problems and persevere in solving them* and *Attend to precision,* respectively. Depending on your state and standards, you may choose other reference documents.

Participants do the task; discussion optional
After providing participants with an introduction, share the task and have them do it. After adequate time is provided, you may have participants share their arguments, or move directly into analyzing the student arguments provided in the next handout. The main purpose of doing the task is to have participants get their heads into the mathematics of the task. It can also allow participants to reflect on the set of arguments produced by their group. If you choose to have participants share out their solution strategies, you might focus the discussion on the similarities and differences among solution strategies. You could also focus the discussion on the degree of generality offered by the argument (e.g., does it demonstrate this claim is true for all consecutive pairs? Some consecutive pairs? If so, which ones?). Particularly as this is an early activity, it is important that the discussion focuses on observations and analysis and not judgements about better, worse, right, wrong. See below for information about the ideas that can be the focus of discussion.

In our experience, if participants are all secondary teachers, there likely will be a preponderance of symbolic approaches used. If there is a mix of grade levels, it is a valuable opportunity to point out the variety of approaches in the room and make sure that non-symbolic approaches are valued, as people tend to see the symbolic as "better" likely because symbolic notation is a tool learned later. Regardless of the approach, a key question is whether the argument compels the claim to be true.

Activity 1.2 Sum of Two Consecutive Numbers Task

Common Issues/Occurrences	**Suggested questions, prompts or points to be raised**
Only symbolic approaches were produced	Ask participants to be generative about other approaches that could be used, or approaches they anticipate their students might use. e.g., How might your students approach this? e.g., How could we solve this problem using a picture or diagram?
One or more participants uses an empirical approach, testing a few	You could ask about why those particular cases were tested. Follow up asking about whether testing those cases ensures

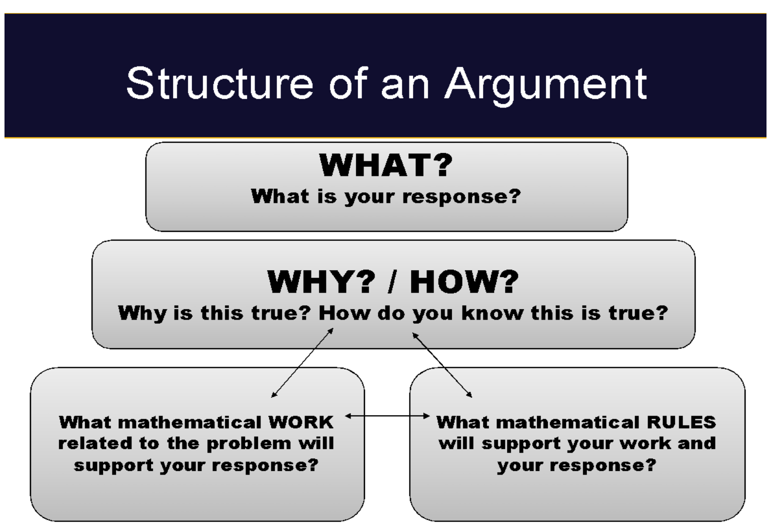
pairs of numbers, which will not demonstrate the claim is true for all cases	the result is true for *all* numbers (and how it indicates that). You may also choose to not address this directly at that time, and rather allow another participant to raise this as a question, or have the issue emerge when student work samples are examined. You might also point out that testing numbers is personally convincing and helps us better understand "how it works" as a set toward developing a more general argument. It is useful work to do, but falls short of being able to demonstrate the claim is true for all consecutive pairs.

Note that there are other Common Issues listed below for facilitating the discussion of student work samples on this task. Those issues may be relevant as well as you facilitate this discussion about their work samples among participants.

Activity 1.3 Student Work Samples: Consecutive Sums Task

Background:

Activity 1.3 provides a context for examining examples of students' mathematical arguments. Prior to having participants analyze the student arguments, what a mathematical argument is and should accomplish is discussed in a little more detail. The slides offer a way to think about the *structure* of an argument and what needs to be included. The following slide excerpted from the draft slides (see below) provides a helpful introduction, highlighting that an argument includes both the "what" – what you are demonstrating is true or not true – and the "why and how" – why it is true (or not) or how you know the claim is true (or not). Note that the Why/How is accomplished by both generating mathematical work in relation to the problem to support the claim as well as drawing on known mathematics (such as rules, definitions, established connections and previous results).



__A NOTE ABOUT ORDER__: You will notice that later in this module, we have found the language *claim, evidence,* and *warrants* helpful for supporting participants to consider the specifics of mathematical arguments. The language comes from Stephen Toulmin's (1958) classic, *The Uses of Argument.* We have, however, varied the point at which we introduce this language to participants. We have found that it can hinder discussion of students' mathematics if introduced too soon. This Facilitator Guide is organized for a later introduction of the specific language, but an early introduction with these general questions. You may choose to introduce this language *prior* to analyzing the student work samples, or *after*, once participants have discussed each sample. Particularly, if you are short on time, you may choose to introduce the language simultaneous with Activity 1.3. Both options are valuable. Please remember that participants may need time and experience to truly make sense of the language/terminology used for the structure. You will likely need to return to the ideas and emphasize that a mathematical argument should be considered holistically, and the concepts *claim*, *evidence*, and *warrants* provide a structure for helping us analyze arguments and understand what makes a strong argument, but is not a check list for students (or teachers) to use.

Set up for the activity: Does the argument show the claim is true?
Provide participants with the set of student work samples (Handout 3: *Consecutive Sums Student Work*). Per the instructions, direct participants to (1) discuss each student's argument, and (2) determine if the argument shows the claim is true. (As desired and indicated above, you can introduce the language of claim, warrant and evidence at this point, or later in the activity.) When discussing the student's argument you might further prompt the group to consider strengths of the student work samples and then areas where they would like the work extended.

Each of the work samples was included strategically to help raise particular points related to mathematical argumentation. In the following section we provide detailed notes for points that participants may raise, or you may wish to raise.[4] We recommend having participants work in pairs or small groups first, and then having a whole-group discussion. To facilitate this discussion, it is helpful to rewrite the sample arguments from Micah, Roland, Angel and Kira onto large chart paper and hang them on the wall. This allows you to directly annotate the sample student work as participants bring up ideas about what they notice and/or to compare across and record key points. (The work samples are also included as slides in the draft powerpoint for reference.)

Please see below for Common Issues that may arise in discussion.

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[4] Note that these are composite work samples, some of which are drawn from a study (Knuth & Sutherland, 2004) and the original activity was developed by the *Justification and Argumentation: Growing Understanding of Algebraic Reasoning* (JAGUAR) Project (NSF , Sean Larsen, PI, DRL 0814829).

Suggestion: We have found it useful to not provide the Kira argument at first, as this approach is common within many groups (particularly those with secondary teachers) and not including it pushes teachers to discuss the other three more fully. Having Kira in the set is useful for demonstrating the variety of approaches and for helping participants think about students over time and how they can represent arguments across grade levels (same claim; different approaches).

Please also see any of the links listed below to view powerpoint slides with audio voice over commentary about each of the work samples.

- **Debriefing Micah**: http://teachers.bridges.education.uconn.edu/wp-content/uploads/sites/1351/2015/09/2a.-Micah-Debrief.mp4
- **Debriefing Angel:** http://teachers.bridges.education.uconn.edu/wp-content/uploads/sites/1351/2015/09/2b.-Angel-Debrief.mp4
- **Debriefing Roland:** http://teachers.bridges.education.uconn.edu/wp-content/uploads/sites/1351/2015/09/2c.-Roland-Debrief.mp4

This above set of narrated slides may also be used directly in your sessions to help debrief this activity. We suggest using these *only after* participants discuss the samples themselves. Additionally, here is a link for narrated slides with summary comments across the work samples: http://teachers.bridges.education.uconn.edu/wp-content/uploads/sites/1351/2015/09/2d.-Summary-Debrief.mp4
Much of the information in these narrated slides is also included in the next section in the discussion of each sample.

What might come up when reviewing this work?

Micah
Micah has done some strong work to get a sense of the problem: Micah tested a range of values (one-digit, two-digit, and four-digit numbers); Micah tested an odd plus an even and also an even plus an odd.

The approach Micah has used is not viable *for demonstrating this claim is true for all consecutive sums* as one cannot look at any finite set of examples and say it is true for the general case. Knowing the result is true for three pairs of consecutive numbers does not mean it must be true for all pairs of consecutive numbers. Note that in their everyday experience, students (and people in general) use this type of inductive reasoning regularly (e.g., for 3 Fridays in a row we had pizza for hot lunch, so this Friday we'll have pizza again). It is a great way to reason in everyday life. In math, however, we need certainty not likelihood to claim something is true. Each additional example might psychologically convince us, but does not add more mathematically to understanding or demonstrating why the claim is true or that the claim is true[5].

Angel

Angel has written a narrative as an argument to show the claim is true, and generally has a logical flow. The approach is viable (sound).

One question often raised is whether the student has said "enough." Is the argument complete, or are there some gaps and we would want to know more? The impetus for this question is in the second paragraph. The approach references what seems to be a *previously established* fact: "And we know that when you add any even number with any odd number the answer is always odd." This point is important to raise. A teacher in a classroom must decide whether a reference like this to previously established knowledge is sufficient, or whether it is another claim that needs to be supported[6]. If indeed the idea that odd + even = odd is well known in the class, this reference to "and we know" may be sufficient. In math, new knowledge is built on previously established knowledge, so this "move" is valid when making an argument. If, however, this "fact" is something the student(s) may or may not know (perhaps they think they know this is true, but it is not a result established fully), then this reference to the type of number that results from adding an odd and an even requires more explanation.

Some participants may also think this statement ("and we know… ") restates the claim to be proven. Note that there is a difference: showing the sum of <u>*two consecutive numbers* is an odd number</u> is different from showing the sum of <u>*an odd and an even* is an odd number</u>. There is mental work done to move from knowing *two consecutive numbers* are being added to knowing *an odd and an even number* are being added. One must apply the definition of consecutive and know something about odd and even numbers. This points to some of the hard work in the area of argumentation where we have to recognize inferences that we (or students) might be making without even knowing it.

Roland

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[5] Note: A common misconception is that all arguments must be general. This is not true. An argument must show the claim is true for *all relevant cases.* If the claim is general, the argument must be general. If the claim is only about one value (the optimal speed is 10 mph) or something not general, then an argument need only address that one solution or those relevant cases.

[6] In some respects, it is the *community* that should decide whether the argument is complete and builds on previously established knowledge. The teacher, however, is the one that must help establish appropriate norms of argumentation as students may not think to question something, or may not feel comfortable doing so. The teacher also is responsible for students' learning more generally, and may wish to "re-establish" this knowledge for pedagogical purposes. The argument and pedagogical purposes should not be conflated however.

Roland has used a pictorial or visual approach. Through the visual, the student offers a way to represent odd and even numbers (as dots in a paired structure [even], and dots paired with one left over [odd]) and a way to add, which here is combining dots.

An important point that is raised by this response is whether the student is showing one example (akin to Micah) or whether this is a general argument, representing the addition an (any) odd and an (any) even.

A strength of this argument is that it (unlike Micah's) **uses properties of even and odd numbers** to create a visual representation. Looking at the work, we might conclude that Roland defines even numbers as "numbers that can be represented by groups of 2s" and odd numbers as "numbers that have one left over (not paired) when you divide it into groups of two." Alternately, one might see Roland's definitions of even as "can be divided equally into two [rows]" and odd as "there's one left over when you try to divide the number into two." The student also **uses general language** from the prompt, such as "a number" and "the next number."

How might this argument be strengthened? Some participants may point out that this argument treats only the case of two consecutive numbers of the form even plus odd, but not odd plus even. A question to consider is: Could the student readily argue that this same logic holds for odd plus even? (In this case we think *yes*, so the approach is sound and needs a slight revision to cover all cases.)
Others may wonder: How could we show this more generally? (which may be phrased as: How could the student indicate that he is thinking about "an even" plus "an odd" and not only 6 plus 7?) Alternately, some may wonder: Can students who do not have mastery of symbolic notation make a general argument? Do they have the tools? If you add ellipses to the middle of each number, as if indicating there's an unspecified number of pairs of dots not drawn, then this argument becomes fully general for evens plus odds, and some language could be added to indicate how this addresses odds plus evens as well.

Kira
Kira used a symbolic approach. In doing so, consecutive numbers are represented generally as n and n+1, and then the sum of these values (2n + 1) is analyzed to argue the result is always an odd value. Note the use of a definition of odd/even ("an odd number leaves a remainder of 1 when divided by 2").

A point to raise in discussion of this work sample alongside the others is the question of how arguments reveal different kinds of understanding (from students) or different mathematical connections. Although Kira's argument is perhaps one that on the surface we think is more sophisticated or more general than the others, we gain less understanding about *why* it is the case that two consecutive numbers always sum to an odd number. (Similarly, Micah's argument is not revealing.) Roland's is perhaps the most revealing, where one can "see" why it is that an odd and an even, when combined, produce an odd. (That one left over or not paired is still left over/not paired.)

An argument or proof that provides insights into *why* a result is true and how the mathematics works, is called an *explanatory* argument or proof (Hanna, 2000). This is an important idea overall, as explanatory arguments are particularly useful for supporting learning.

Overview of common issues and key ideas in student work samples

The left-hand column of the following table provides some common discussion points and issues that come up for participants. You notice that there are repeats from some of the ideas from the above discussion, but this organization might be a helpful reference sheet for you to use while facilitating the activity. The right-hand column offers some suggestions for discussion.

Common Issues: Activity 1.3	Suggested questions and prompts
Focus on correctness It is important for teachers to focus on a holistic sense of the arguments students produce. A viable argument is one that provides a sequence of statements and reasons that demonstrate a claim is true or false. Participants may focus overly on the mathematical correctness of an argument, as computational accuracy is strongly emphasized in our schooling. This can be problematic because a mathematically accurate solution does not imply that the student has necessarily constructed a viable argument. Conversely, a fairly strong mathematical argument may have minor errors that are easy to shore up, and it's important to not have a minor error overshadow a strong approach to an argument.	What claim(s) did the student make? How do they defend that claim? Does the student back up their ideas with reasoning? Set aside for the moment the computational or spelling errors. Suppose these were corrected, how would you evaluate the reasoning the student presents? Let's compare Micah (empirical) to Roland (visual, shows structure of odd and even). Can Micah continue to develop this approach and have an argument that shows the claim is true? Can Roland continue to develop this approach and show the claim is true?
Misconception that Examples are Enough (Micah) Students, as well as teachers, often begin their work on the Sum of Two Consecutive Numbers Task by experimenting with the sums of various example numbers. (See Micah's Sample Work). This approach is useful for getting a	What allows you to be convinced by three examples? Do you think students would be convinced by three examples? Would 4 be enough? Would 5? Would any three examples convince you (e.g., 1+2, 2+3, and 3+4)? What's useful about which

"feel" for the numbers, or convincing oneself that the claim is likely true. The use of any specific set of examples alone cannot demonstrate that a claim related to a general case (*all* numbers) is true. In order to be a viable argument, for this task, the argument must include reasoning about *any* two consecutive numbers.	examples the student chose? What is still not known? In everyday life, we often use *inductive reasoning* to look at examples and infer a general idea. In pure math, however, we need to use *deductive reasoning.* No set of examples – even 2000 examples – can show a claim is true for all numbers (or pairs of consecutive numbers).
Angel's argument is circular (misconception) Some participants may think "And we know that when you add any even number with any odd number the answer is always odd" is assuming to be true what you need to show is true.	Is there a difference between adding an even number and odd number versus adding two consecutive numbers?
Roland's is general; Roland's is just an example Some participants may see this as an example of adding 6 + 7.	Ask participants to talk about what they notice in Roland's argument that leads them to think Roland is focused on one example and what they notice that leads them to think Roland is thinking about pairs of consecutive numbers more generally. You might add ellipses to Roland's diagrams between pairs of dots. Ask participants if that makes it general. Ask participants how they could show odds and evens more generally *without* using symbolic notation. The point here may be that, given the tools available, students may not be able to express an idea completely generally with precision. We need to look at their tools available and the language to see whether they are reasoning about one example or more generally. This also points to the power of algebra. We need algebra to help us do this general work (but can still ask questions about general claims even if students have not mastered symbolic notation yet!)
Symbolic is "best" (Kira) In comments, some participants may explicitly or implicitly indicate that the best type of argument uses symbolic notation.	What is "better" or "stronger" about Kira's argument?

Although true that symbolic notation helps us communicate mathematically with precision (and in fact, that was a huge impetus for its development), expressing ideas symbolically is one skill, and making sense of a proposition and being able to explain why a result is true is equally important. This brings us to a distinction between focusing on skill in writing or expressing an argument vs using argumentation as a way to help develop (and assess) students' understandings of mathematical ideas.	How does the symbolic notation help the student make the argument general? How does this differ from Roland's visual and effort to express ideas generally? (e.g., generally represent two consecutive numbers) Looking across the students, who do you think (and based on what evidence) has the strongest grasp of *why* it must be the case that the sum of two consecutive numbers always is an odd number (it cannot be an even number, no matter what)?

Activity 1.4 Analyze Components of Arguments

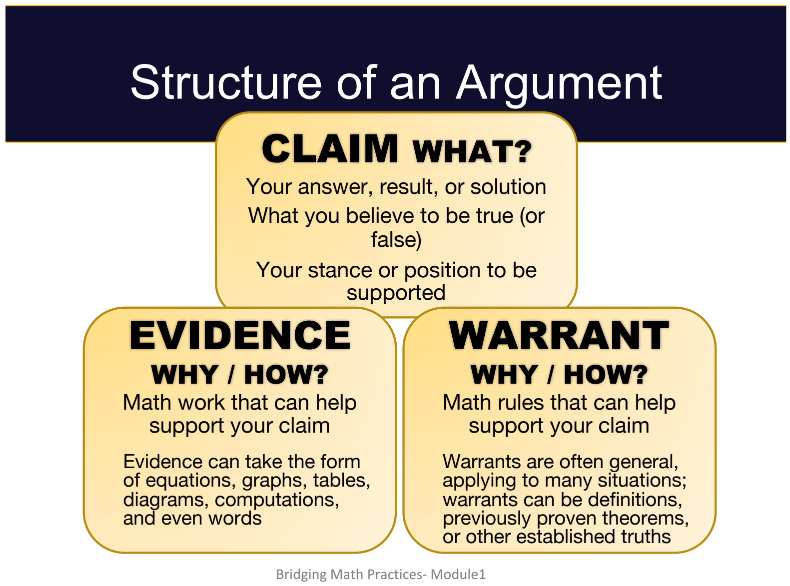
Participants are introduced to the language *claim, evidence,* and *warrants*, as a way to make sense of the structure and components of a mathematical argument. We leverage Toulmin's work to help to develop some common language to discuss and analyze arguments, specifically, *claim, warrants* and *evidence*. In addition, this activity allows participants to pinpoint issues they were noticing and described in Activity 1.3 but perhaps did not have clear language to yet describe.

Building on the idea that a mathematical argument includes a claim as well as an explication of how one knows that the claim is true (or false), we identify that this explication involves both mathematical work, or *evidence,* that is local to the problem/question, and mathematical rules, or *warrants,* that are true across situations and contexts, whose applicability can be shown to the given situation To illustrate these ideas and connections, we include slides in which we link the definition of each term to guiding questions. Below we list the definition of each term and a corresponding guiding question.

Term	Definition	Guiding Question
Claim	Your answer, result, or solution; What you believe to be true (or false); Your stance or mathematical position to be supported	What is your response?
Evidence	Math work that can help support your claim. Evidence can take the form of equations, graphs, tables, diagrams, computations, and even words.	What mathematical work related to the problem will support your response?
Warrants	Math rules that can help show how your evidence supports your claim.	What mathematical rules, definitions, or previously established facts will

	Warrants are often general, applying to many situations. Warrants can be definitions, previously proven theorems, or other established truths.	support your work and your response?

This slide shows the relationship between the previously introduced set of questions related to an argument and this terminology (slide 22).



We suggest that the group then works through an example argument (Micah) together to help participants further make sense of the language and structure of an argument. Participants revisit and analyze the four work samples in response to the Sum of Consecutive Numbers prompt. Participants can use the new language to describe more precisely the arguments and comment on strengths and weaknesses. For example, participants can now note that Angel used the fact that the sum of an odd and even was an odd number as a *warrant*, but that itself may also need to be proven (and was potentially an unsubstantiated warrant). Below are notes that describe each student response in relation to these three elements.

Note on Warrants: We have found that *warrants* are often the most difficult aspect of this structure for participants to understand. In the **Additional Materials section** at the end of this guide, we include links to a series of narrated slides that provide further explanation and examples related to the structure of arguments with an emphasis on warrants. You may find the second narration on Toulmin's structure for arguments particularly useful for making sense of *warrants.* Further Analyzing Arguments: Toulmin's and a Focus on Warrants. (https://drive.google.com/open?id=0B4mQL9do5xayQVV2OVNhTHVvcUE)

Micah:
Claim: it's true
Warrant: if a result holds for three examples, it holds for all numbers
Evidence: three examples worked out
Note that the claim is true, but rests on a faulty argument. The evidence is accurate, but the warrant is not valid. The warrant is not made explicit (one might call this an implicit warrant, or unstated warrant). In mathematics, three examples can help us conjecture a result may be true, but it does not demonstrate the result to be true.[7]

Angel:
Claim: it's true
Warrants: We know from a previously established fact that adding an odd and an even makes an odd. We know that you always have an odd and an even when you have two consecutive numbers.
Evidence: none
Note that each of the warrants could be taken at face value (considered previously established and agreed upon by the community), or each could be considered a fact that also needs to be shown to be true: Why does an odd plus an even make an odd? How do we know that consecutive numbers are always one odd and one even?

Roland:
Claim: it's true
Warrants: The warrants here are implicit. Roland does not explicitly state a definition of even number, but represents an even number as two rows of dots, where all dots have a partner. Similarly, odds are represented as two rows of dots where one dot has no partner. [Note that here we are assuming Roland is thinking more generally and not just about the numbers 6 and 7.]
Evidence: Visual representation of an even and an odd, which combined shows an odd ("always one leftover").

Kira:
Claim: States that 2n+1 "is always an odd number" which affirms the claim (as this is the sum of the two consecutive numbers).
Warrant: The main warrant rests on a definition of odd, which is "leaves a remainder of 1 when you divide by 2." The sum, 2n+1, is determined to be odd based on this definition.
Evidence: The evidence provided is showing the sum of n and n +1 is $2n$+1, and that the sum leaves a remainder of 1 when divided by 2, by adding the parenthetical information that (2n+1)/2 = n remainder 1.

[7] Note the contrast with science: in science, a preponderance of evidence (and no compelling counter examples), along with theoretical backing, is enough to make something considered a working truth. This is quite different from mathematics.

One can always follow up asking for more evidence or more support for any piece of information (e.g., Can you explain how you know n and n + 1 represent consecutive numbers? Can you explain how you know 2 goes into 2n+1 n times with a remainder of 1? Those are details which may or may not need to be included depending on the community/audience).

Key point for Activity 1.4

The language of claim, warrant and evidence is meant to help with some analysis and conversations. Getting too into-the-weeds with the language is likely not helpful at this point. Note that most warrants (that are not definitions or axioms) could be considered claims themselves, that also need support, or at some point needed support to show they were true. There is a judgment call involved in deciding an argument is "complete enough" or has made explicit all that needs to be made explicit. It is also not uncommon for teachers to "fill in the warrant" mentally, even when the student does not articulate it. That can be a point of learning for participants, as they notice the importance of having students articulate those connections (both for their learning and for the purposes of developing their skills for argumentation).

Overall, it is less important that participants have specific names for specific pieces of a student's argument, and more important that they can identify the chain of reasoning (the logic), what "support" is being offered for the argument, and where there might be gaps or missteps. It is also important for participants to develop a sense of what an argument *is not,* and tune their thinking to focus on the relevant components.

Activity 1.5 Bridging to Practice

As described in the Overview of the Facilitation Guides, the Bridging-to-Practice activities are a staple of this professional development that support participants to connect the concepts of the PD with their work in classrooms and schools. We first offer ideas for the Monthly Format and then ideas for the Workshop Format.

Monthly PLC Format

For the PLC format, we encourage you to design activities that support participants to: (a) continue to think about the ideas already presented, (b) try out some ideas in a classroom setting with students, and/or (c) seed ideas for discussion in subsequent sessions.

Ideas:

One option for a Bridging-to-Practice activity between Module 1 and Module 2 is for participants to select one or two tasks that they thought prompted argumentation. They also might consider tasks that they would *like* to revise to prompt argumentation. This activity provides opportunities for participants to closely examine the curricular materials they use at the grade(s) they teach with an eye towards mathematical argumentation.

Although mathematical tasks are not a primary focus of Module 1, tasks play an important role in Module 2.

Another option is to have participants do a formative activity where they pose argumentation prompt to their class and then review the student responses. Akin to the work they did on the Sum of Two Consecutive Numbers task, they can notice the different approaches student took, the degree to which students offered a viable argument, and areas of strength and improvement for their class. Some areas for improvement may be linked to developing a better understanding of what an argument *is,* whereas other areas will be related to communicating the argument or making sense of the content.

Potentially in conjunction with either of the above, a third potential Bridging-to-Practice activity for between Modules 1 and 2 is to ask participants to survey their students about their knowledge of mathematical argumentation. Since participants may have differing levels of knowledge of mathematical argumentation, this activity supports all participants to consider the ideas of Module 1 through the perspective of their students. This activity also helps participants get a sense of what students do and do not know about mathematical argumentation. Below are some possible questions to include on such a survey:

SURVEY on MATHEMATICAL ARGUMENTATION

1. In your own words, what is a mathematical argument?

2. Why is argumentation particularly important in mathematics?

3. Approximately, how often do you construct viable arguments in your math class? Circle one.

   Every day   3-4 times per week   1-2 times per week   Monthly   Infrequently

   Comment (optional):

4. Approximately, how often do you revise your own reasoning, or critique the reasoning of others in your math class? Circle one.

   Every day   3-4 times per week   1-2 times per week   Monthly   Infrequently

   Comment (optional):

5. Consider the statement: *When you add any 2 even numbers, your answer is always even*.

   Is this statement true? Decide whether it is true or not and offer a mathematical argument to support your answer)?

6. (Optional – suggested for secondary only) You are building a sequence of geometric figures with toothpicks, by following a specific pattern (making triangles up and down alternatively). Here are pictures of the first three figures you build.

t=1 t=2 t=3

   a. Draw the 4th figure in the sequence (t = 4).

   b. How many toothpicks will you need to build the 20th figure (t = 20)? Show how you know.

   c. How many toothpicks will you need to build the figure *t*? You may describe how you would find the value or write a formula. **Be sure to provide an argument to show that your answer is correct.**

NOTE: You might also find it informative to ask students to critique one of the student work samples from the Module (e.g., Roland's argument).

Workshop Format – Student Work Sample Sorting Activity

For the five-day workshop format, we outline a 70-minute activity to provide the opportunity to engage these ideas in ways that are directly applicable to practice, even when the participants may not be teaching at the time they do the modules. For this Bridging-to-Practice activity, participants work in small groups (3-4 people) as if part of a professional learning community (PLC) or as members of a department to examine, discuss, and sort samples of student work based on the quality of students' arguments. The group's work together is guided by a protocol. The one used for this activity is in the general category of what is known as a "tuning protocol" as it allows colleagues to "tune" to one another's ways of thinking about student work and what counts as a quality mathematical argument.

The Bridging Math Practices website has materials for tasks/sets of work samples to choose from. These are part of our Argumentation Resource Packets (ARPs). These focal tasks were implemented by teachers in our 2014-2015 cohort, and the student work samples are from their students. The samples were culled and put together as a packet by grade-level teams of teachers and coaches in the cohort. The full set of ARPs can be found at http://bridges.education.uconn.edu/argumentation-resource-packets-2/. (There is also a link to this section of the website at the bottom of the list of handouts for Module 1.)

In our implementations, we have used the student work samples from the “DJ Problem” as for secondary teachers (http://bridges.education.uconn.edu/wp-content/uploads/sites/753/2016/03/DJ-Prom-Alg-I-ARP-Student-Work-Sorting-Packet.pdf). The elementary packet we used includes work samples for a task called “Is it 1/4?” (http://bridges.education.uconn.edu/wp-content/uploads/sites/753/2016/03/LauraSaysOneFourthIsShaded-Grade-4-ARP-Student-Work-Sorting-Packet.pdf).
Other Sorting Packets can be used as well. Please see the paragraph at the end of this section for more information.

Set Up: We recommend about 10 minutes for setting up this activity, including getting participants into small groups, explaining the Student Work Sorting Protocol, and distributing materials. If this is the first time participants have used a protocol, you may want to expand this introduction beyond 10 minutes.[8] A helpful reference is http://edglossary.org/protocols/.

Give each participant one complete, stapled work sample packet and a copy of the Bridging_Student-Work-Sorting-Protocol handout. Additionally, each group should be given a group set of the appropriate sorting packet of student work samples. We recommend that the group set be copied onto color paper and left unstapled to allow for participants to place the samples in different piles.

Sorting Protocol: The protocol provided, as written, is approximately 35 minutes for small groups to work together, and then an additional 15 minutes for a full group discussion. Read the protocol carefully before implementing this activity. Participants first spend time working individually with the student work samples from his or her own packet and decide which of the following three categories is most appropriate for each student’s argument: (1) High Quality Argument, (2) Adequate Quality Argument, and (3) Low Quality Argument. Note that these labels are not offered with a strict definition. One of the purposes of this activity is to help participants become aware of and reflect on what they are noticing and valuing as they review student arguments. Two participants might notice the same strengths and weaknesses in the student work samples, but “weigh” these differently, and consequently come to different determinations about the overall quality of the argument. You may wish to put participants’ minds at ease by explaining this purpose up front and noting that they are not being provided with definitions or checklists of criteria for these

[8] An excellent book on protocols is *The Power of Protocols: An Educator’s Guide to Better Practice* by J. McDonald, N., Mohr, A. Dichter, and E. McDonald (2003).

levels. [Note that the Sorting Protocol used for this module is a modification of the one posted with the ARPs.]

After participants sort individually, provide them with the "group packet" of student work samples (on colored paper, not stapled), and have them work collaboratively in small groups to come to a consensus with a group sort. The Student Work Sorting Protocol handout explains that during this phase, participants should assume the roles of Recorder and Handler to help delegate the work and to encourage collaboration throughout the activity.

Once the small groups have had time to come to consensus on their sorting and discuss their reasoning behind their choices, bring the groups together for a discussion. The small group discussion likely helped participants begin to articulate what they are noticing and how they are valuing it, and the whole group discussion can further advance this goal.

Whole-Group Discussion: We recommend about 15 minutes for whole-group discussion of their different "sorts" and thoughts about the student work samples, though this discussion can easily take more time.

One way to facilitate this is to have all groups that sorted the same set of student work samples (same task) write their "final sorts" in a chart/table that is posted on the board, chart paper, or projected. As the facilitator, you can then see where there are differences and overlap in how groups evaluated the student arguments and those points of difference, as well as commonality, can be raised for discussion. You can also open the floor for general discussion and questions.

As an example, with the DJ sorting task, we have found that Student 5 often generates a lot of discussion, as many participants find they do not "follow" the student's chain of reasoning at first, but then are able to make sense of it and find it is a strong argument. Student 10 often prompts comments about how the student did a lot of wonderful math, but it's not clear that the student knows s/he could have provided just one of those representations and explained it well and had an equally (or more) strong argument. It's also not always clear if the student knows how the work is supporting the claim and articulating the connection.

The discussion of the student work samples should keep participants reflecting on what they are identifying as characteristics of a *high quality argument*. Participants also need to consider the arguments holistically.

Below is a list of some key points that participants might take away for this activity:

- Determining the quality of a students' written argument is not a fully "cut and dry" process.

- In order to determine whether an argument is of high quality, one must attend to the argument as a whole. The ways the individual parts of an argument (i.e., claim, evidence, warrants) are connected is important to the overall quality.
- Different representations can all be valuable in their own right. Not only symbolic representations or lengthy written explanations can make for strong, viable arguments.
- When thinking about argumentation, one's mind has to be open to the argument the student is producing. We, as teachers or coaches, might have a preferred way of approaching the problem, but there is no one right way. As long as the argument makes sense and compels the claim, it can be a strong argument and it does not matter if the student used the concept taught in this unit, last unit, or last year.
- Argumentation by nature cuts across math topics and/or allows students to draw on whatever tools make the most sense to them in relation to that problem (and articulate that relationship)
- It's important to "look past" features such as neat penmanship and organization, the use of good math vocabulary, complete sentences, or requiring symbolic representations. These features may help support the communication of the student's ideas, but these are not reasons why an argument is or is not a strong *argument*. (e.g., Not using the work denominator by itself doesn't lessen the quality of the argument, although the teacher may be pleased when a student uses such terms.) These "additional features" are important to acknowledge, however, and attention to elements like vocabulary may be critical for some arguments for students to effectively and precisely communicate their ideas.

You may wish to close the activity with some kind of synthesizing or reflective question. One option is to ask participants for key take aways and record these publically (chart paper or in a powerpoint slide). You might also consider an exit slip, a pair-share with some specific prompt, or asking for questions they hope are addressed in future discussions. (Note that some of these ideas can be used for Session Closure as well, and not just to wrap up this Bridging-to-Practice Activity).

Argumentation Resource Packet (ARP) materials on the website
We include here some additional information about the ARPs on the website.
The website has 7 tasks, with 5-10 student work samples each, at grade levels 3, 4 (two packets), 5, 3-6, Geometry (HS) and Algebra I (HS). The accompanying materials include the task, the work samples, a sorting protocol (different from the one used for these PD modules that is more catered to PLC work), and commentary on each of the work samples that was developed by our 2014-2015 cohort. Read more at
http://bridges.education.uconn.edu/about-arp/

Activity 1.6 Session Closure

The purpose of the Session Closure is to provide some summary of the big ideas teachers from the module and solidify connections to practice. Some of this time could also be used to get feedback from participants in order to gain information about (a) what they are understanding from the material and (b) how the facilitation, session organization, etc., are working for participants.

For the workshop format, please request that participants bring one or two tasks that they think prompt argumentation to the next session. They also might consider tasks that they would *like* to revise to prompt argumentation. In Module 2, the Bridging-to-Practice activity provides opportunities for participants to closely examine the curricular materials they use at the grade(s) they teach with an eye towards mathematical argumentation. Alternately, you can provide participants with tasks from their curriculum or related to topics they teach.

Additional Resources: Module 1

Articles and videos on mathematical argumentation, justification and proof/proving.

Cioe, M., King, S., Ostien, D., Pansa, N., & Staples, M. (2015). Moving Students to "the Why?" *Mathematics Teaching in the Middle School, 20*(8), 484–491.

McDonald, J., Mohr, N., Dichter, A. & McDonald, E. (2003). *The power of protocols: An educator's guide to better practice*. New York: Teachers College Press.

Mejía-Ramos, J. P., & Inglis, M. (2009). Argumentative and proving activities in mathematics education research. In F. Lin & F. Hsieh (Eds.). *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education* (Vol. 2, pp. 88-93), Taipei, Taiwan. Retrieved from http://140.122.140.1/~icmi19/files/Volume_2.pdf

Monte Python's *Argument Clinic*.
https://www.youtube.com/watch?v=kQFKtI6gn9Y&feature=kp

We have used the first 3:50, but do not play 0:38 – 1:16 (which is not relevant and involves some questionable language). It is interesting to note that the actors are arguing about what counts as arguing. Two quotes worth noting are:
1:55 "This isn't an argument - it's just contradiction!"
2:15 "An argument's a collective series of statements to establish a proposition."

Olmstead, E. A., (2007). Proof for Everyone. *The Mathematics Teacher, 100*(6), 436–439.

Otten, S., Herbel-Eisenmann, B. A., & Males, L. M. (2010). Proof in algebra: Reasoning beyond examples. *The Mathematics Teacher, 103*(7), 514–518. Retrieved from http://www.jstor.org/stable/20876681

Stylianides, A. J. (2007). Proof and Proving in School Mathematics. *Journal for Research in Mathematics Education, 38*(3), 289–321. Retrieved from http://www.jstor.org/stable/30034869

Self-paced PD module: A sequence of three web-based self-paced modules are available. The first of these on What is a Mathematical Argument? discusses many of the same ideas presented in this module. This online module is well suited for individuals to work through or for pairs. It could provide a way for someone to "catch up" with a group if they have missed a session. It could also be used as supplementary material for this module, or as a means to reinforce or refresh the ideas. http://teachers.bridges.education.uconn.edu

Additional Resources to Support the Structure of Arguments, Particularly Warrants

If facilitators or participants want to dig in deeper with mathematical argumentation, or have additional opportunities to make sense of these ideas, they may find the following set of narrated slides useful. Please *download* and then use "Play from Start" under the SlideShow menu in order to hear the narration. The four narrations are in this folder https://drive.google.com/folderview?id=0B4mQL9do5xayM3RGSUJBS1RjaDg&usp=sharing and a link to each narration is provided here as well.

1. **Argumentation Introduction**: [2:05] This narration provides a basic overview of what a mathematical argument is.
   https://drive.google.com/open?id=0B4mQL9do5xayZnRtMUhhdWRXdlE

2. **Argumentation: Toulmin and Warrants** [8:05]
   This narration provides an introduction to Toulmin's model and language for arguments: *claim, evidence,* and *warrants*, which is then applied to three examples (Micah, an everyday example, Angel). Extra attention is given to the concept of *warrants*.
   https://drive.google.com/open?id=0B4mQL9do5xayQVV2OVNhTHVvcUE

3. **Argumentation: Focus on Procedural** [13:05]
   This narration gives a closer look at how problems that seem to be more procedural or computational and involve only "showing your steps" are opportunities for argumentation. Sharing warrants/reasons can turn an explanation of steps into an argument.
   https://drive.google.com/open?id=0B4mQL9do5xayeTJYajdOZnZDQzg

4. **Argumentation – The Warrant is Missing!** [3:06]
   This narration uses every day examples to help clarify warrants.
   https://drive.google.com/open?id=0B4mQL9do5xaycE85Mzd5cXY0V0k

References: Module 1

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics, 44*, 5-23.

Knuth, E., & Sutherland, J. (2004). Student understanding of generality. In D. McDougall & J. Ross (Eds.), *Proceedings of the 26th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Toronto, Canada: University of Ontario* (Vol 2., pp 561-568).

Toulmin, S.E. (1958). *The uses of argument*. Cambridge: Cambridge University Press.

Module 1 Facilitation Guide prepared by Megan Staples and Jillian Cavanna based on 2014-2015 and 2016 implementations of the Bridging Math Practices Project. Last updated September, 2016.