Chapter 7-lntegration Techniques

§7.8 - Introduction to Differential Equations

In this section we will examine Differential Equations (DE) that are used in engineering, economics, finance, and to model population growth in the natural and biological sciences. Differential equations represent one of the most important applications of the Calculus that is, the modeling of real-world phenomenon. A differential equation is an equation that contains an unknown function and some of its derivatives or simply put a differential equation is an equation containing derivatives. Some simple examples of differential equations are dy = 2x , dx

dx = 3y3 '

d 2 y

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dx 2

= 4x + 3 . The "order" of a differential equation is the order of the derivative of the highest

order that appears in the equation. For example, dy = 2x is a first-order differential equation

dx

while d

2

;' = 4 x+3 is a second-order differential equation. The simplest type of DE is a first-

dx

order equation of the form = f ( x ).

Since a DE is an equation involving derivatives solving a DE requires inte!,rration, that is, you must undo the derivative y'(t) in order to fmd y(t). As you know, integration involves the introduction of an arbitrary constant, so the general solution of a first-order DE involves one arbitrary constant while a second-order DE involves two arbitrary constants. Overall, an

nth- order DE involves n arbitrary constants. Sometimes an initial value or an initial condition

is given to a DE. This initial value makes it possible to determine the specific values of the arbitrary constants. Let's begin by examining some examples of differential equations.

Example I

Solve the initial value problem y'(t) = 10e-tfz, y(O) = 4

In general, if y(t) is the value of a quantity at timet and if the rate of change of

y with respect to t is proportional to its size y( t) at any time, then

y'(t) = (k)(y(t))where k is a constant. Note that if k is positive, then the population increases

(Law of Natural Growth); if k is negative, the population decreases (Law of Natural Decay). Therefore, these functions satisfy a first-order DE of the form y'(t) = (k)(y(t)) where k is a real number. The general solution to this DE is given by the function y( t) = Cekt where C is an arbitrary constant.

Next, we consider the first-order linear equation y'(t) = (k)(y(t)) + b where k and bare real numbers. Let's analyze this DE and find its solution. The terms of the cquatin have the

following meaning:

y'( t)

Rate of change of y

(k)(y(t)) natural growth or decay rate of y

+ b

growth or decay rate

due to external effects

For example, ify represents the amount of a drug in the blood, (k)(y(t)) (with k < 0) models exponential decay of the drug through the kidneys, and b > 0 is the rate at which the drug is added to the blood intravenously. The solution to this DE is given next.

Definition

The general solution of the first-order equation y' (t) = (k)(y(t)) + b where k and bare real

numbers, is y(t) = C e kt -!!., where Cis an arbitrary constant. Given an initial condition, the

k

value of C may be determined.

Example 2

Verifythaty(t) = Cek t - isasolutionofy'(t) = (k)(y(t)) + b

A drug is administered to a patient through an intravenous line at a rate of 6 mg. The drug has a

hr

half-life that corresponds to a rate constant of o.o3. Let y(t) be the amount of drug in the blood

hr

for t :=:: 0. Solve the following initial value problem and interpret the solution.

DE: y'(t) = -0.03y(t) + 6

Initial condition:y(O) = 0

Separable First-Order Differential Equations

The most general first-order DE has the form y'(t) = F(t,y), where F is a given function that my involve both variables. We may be able to solve such an equation provided we can rewrite it in the form g(y)y'(t ) = h(t), where the terms involving y appear on one side ofthe equation separated from the terms that involve t.An equation that can be written in this form is daid to be separable. We can solve the separable equation g(y)y'(t) = h(t) as follows:

Find the function that satisfies the following initial value problem: dy = y 2 e -x, y(O) =

dx z

Example 5

Fifty fruit flies are in a container at the beginning of an experiment. Let P(t) be the number of fruit flies in the container t days later. At first, the population grows exponentially, but due to limited space and food supply, the growth rate decreases and the population is prevented from growing without bound. This experiment can be modeled by the logistic equation

dPt = O.lP (1 -

P)

300

Together with the initial condition P(O) = 50. Solve this initial value problem.

Direction Fields

Suppose you are asked to sketch the graph of the solution ofthe initial-value problem

y' = x +y y(O) = 1

The equation y' = x + y tells us that that the slope at any point (x,y) on the graph is equal to the sum of the x - and y - coordinates of the point. In particular, because the curve passes through the point (0,1) (see the initial value above), its slope at that point must be 0 + 1 = 1. So we can show this slope by drawing a small line segment through (0,1) with slope 1 (see figure

below).

)'

(0, I )

Slope at (0, I)

isO+I=l.

0 X

If we continue on in this manner by drawing short line segments at a number of points (x, y)

with slope x + y the result is called a direction field and is given in the figure below.

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Next we can draw the solution curve to y' = x + y through the point (0,1) by following the

direction field as in the figure below.

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Let's look at some examples.

Example 5

Sketch the direction field for the first-order linear equation y'(t) = 3y -6

y

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y=2isao equilibrium solution

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(Direction field for y'(t) ""' 3y - 6)

Consider the Logistic Equation, that is, dP = O.lP (1- ..!:..) fort ;::: 0 and its direction field

dt 300

(see figure below). Sketch the solution curves corresponding to the initial conditions y(O) = SO,

y(O) = 150, and y(O) = 350.

p

(P' < OforP > 300)

Carrying

capacity P = 300

200

tOO

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0 20 40 60

Solution curves in the direction field

for dPi = O.hn\11 -

P )

300